SECOND ORDER LINEAR DE.

an equation which has the general form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0(x)y = g(x)$$

If the above DE is solved subject to initial conditions given by

$$y(x_0) = y_0,\ y'(x_0) = y'_0,\ \dots\ y^n(x_0) = y^n_0$$

Then it is called an initial value problem (IVP)

In the case when $n=2$, the above reduces to 2^{nd} order linear DE given below.

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

for $y(x_0) = y_0$ & $y'(x_0) = y'_0$, the solution is a function defined on an interval I whose graph passes through (x_0, y_0) such that the slope of the slope curve at $x_0 = y'_0$

EX: A function

$$y = 3e^{2x} + e^{-2x} - 3x$$

is the solution to

$$\frac{d^2y}{dx^2} - 4y = 12x,\ y(0) = 4 \ \ \&\ \ y'(0) = 1$$

<u>PROOF</u> $\quad \frac{d^2y}{dx^2} = 12e^{2x} + 4e^{-2x}$

$$\frac{d^2y}{dx^2} - 4y = 12e^{2x} + 4e^{-2x} - 4(3e^{2x} + e^{-2x} - 3x)$$

$$\therefore \frac{d^2y}{dx^2} - 4y = 12x$$

BOUNDRY VALUE PROBLEM

Another type of problem consists of solving a DE of 2nd order which the dependant variable y and it's durivatives are specified at different pts.

EX.

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

subject to $y(a) = y_0$; $y'(b) = y_0$

this is called a boundry problem. other parts of the boundry may be:

$$y'(a) = y'_0 \quad ; \quad y(b) = y_1$$

$$y(a) = y_0 \quad ; \quad y'(b) = y'_b$$

$$y'(a) = y'_0 \quad ; \quad y'(b) = y'_b$$

EX. the following is a boundry value problem

$$x^2y'' - 2xy' + 2y = 6$$

subject to $y(1) = 0$ & $y(2) = 3$

THE D OPERATOR

If D is an operator, then D^n is an operator indicating n successive applications:

$D = \frac{d}{dx} = D_x = D$ } Drop the x.

EX. $D(e^{2x}) = 2e^{2x}$

$$D^2(e^{2x}) = 4e^{2x}$$

EX

$$\frac{d}{dx}\left[\sin x\right] = D\sin x = \cos x$$

EX.

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$$

we may writer

$$D^2y + 3Dy + 2y = \sin x$$

or

$$y\underbrace{\left[D^2 + 3D + 2\right]} = \sin x$$

known as the differential opperator and given by the the letter "L"

so we may write the above equation

$$Ly = \sin(x)$$

where

$$L = D^2 + 3D + 2 = P(D)$$

In general we may write

$$Ly(x) = r(x)$$

this abrivicted as

$$Ly = r$$

L is a linear opperator

$$L(u+v) = Lu + Lv$$

where u & v are functions of x.

$$L(au) = aL(u)$$

where a is constant

now we consider when $r(x) = 0$ for

$$Ly(x) = r(x) = 0$$

then we consider it a homogenous equation. when $r(x) \neq 0$, equation is known as non-homogeneous.

EX.

$$2y'' + 3y' - 5y = 0 \quad \Big\} \text{ homogenous. DE}$$

EX.

$$x^3y''' + 2xy'' + 5y' + 6y = e^x \quad \Big\} \text{ nonhomogeneous. DE}$$

PRINCIPLE OF SUPER POSITION. (HOMOGENOUS EQU)

homogeneous differential equation $Ly = 0$

a homogeneous linear 2^{nd} order equation is written as.

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

If $a_2(x), a_1(x), a_0(x)$ are constants. and if y_1 & y_2 are the solutions to the 2nd linear DE with constant coeffients, then the general solution to the DE

$$Ly = 0$$

y is a linear combination written as

$$y = C_1y_1 + C_2y_2$$

start with

$$y = C_1y_1 + C_2y_2$$

$$y' = C_1y_1' + C_2y_2'$$

$$y'' = C_1y_1'' + C_2y_2''$$

now we go back to the general equation, we get

$$a_2(x)\left[C_1 y_1'' + C_2 y''\right] + a_1(x)\left[C_1 y_1' + C_2 y_2'\right] + a_0(x)\left[C_1 y_1 + C_2 y_2\right] = 0$$

now expand.

$$a_2(x)C_1y_1'' + a_2(x)C_2y_2'' + a_1(x)C_1y_1' + a_1(x)C_2y_2' + a_0(x)C_1y_1 + a_0(x)C_2y_2 = 0$$

group C values

$$C_1\left[a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y\right] + C_2\left[a_2(x)y_2'' + a_1(x)y_2' + a_0(x)y_2\right] = 0$$

now we can see that when $y_1 \neq y_2$; $y = C_1 y_1$ and $y = C_2 y_2$ satisfy the DE.

$$y = C_1(0) + C_2(0) = 0$$

EX.

$$\frac{d^2y}{dx^2} + y = 0$$

$$\left[D^2 + 1\right]y = 0$$

$$Ly = 0$$

later we will see that

$$y_1 = \cos x$$

$$y_2 = \sin x$$

and the general solution

$$y = C_1 \cos x + C_2 \sin x$$

EX.

$y_1 = e^x$ & $y_2 = e^{2x}$ & $y_3 = e^{3x}$

are solutions.

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

LINEAR INDEPENDENCE

"THE WRONSKIAN"

consider the interval I where y_1 & y_2 are functions of x, then y_1 & y_2 are said to be linearly independent. if the Wronskian

given by $$w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$\underbrace{y_1 \cdot y_2' - y_2 \cdot y_1' \neq 0}_{\text{Determinit.}}$$

If $w(x) = 0$, then they are linearly dependent.

EX.

let $y_1 = \sin x$ & $y_2 = \cos x$

now we assume that $y_1 = \sin x$ & $y_2 = \cos x$ are the solutions to

$Ly = 0$

so we check the determinate.

$$w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= (\sin x)(-\sin x) - (\cos x)(\cos x)$$

$$= -\sin^2 x - \cos^2 x = -1 \neq 0$$

$\therefore$ they are linearly independent.

EX,

Are y_1 & y_2 given below linearly independant?

$$y_1 = \sin^2 x\ , \quad y_2 = 2 - 2\cos^2 x$$

$$w(x) = \begin{vmatrix} \sin^2 x & 2-2\cos^2 x \\ 2\sin x\cos x & +4\cos x\sin x \end{vmatrix}$$

$$= (\sin^2 x)(4\cos x\sin x) - (2-2\cos^2 x)(2\sin x\cos x$$

$$= 4\cos x\sin^3 x - 4\sin\cos x + 2\cos^3 x\sin x$$

$$= 4\cos x\sin x(\sin^2 x - 1 + \cos^2 x)$$

$\therefore$ y_1 & y_2 are linearly independant.

$$Ly(x) = 0$$

For ex,

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 0$$

$$\underbrace{[2D^2 + D + 4]}_{}y = 0$$

$$Ly = 0$$

COMPLETE SOLUTION:

of homogeneous and non homogeneous DE.
consider the linear first order DE given by

$$D^2y + PDy + qy = F(x) \qquad D = \frac{d}{dx}$$

for $f(x) = 0$, the answers are y_1 & y_2, and the complete solution is

$$y = C_1 y_1 + C_2 y_2$$

the complete solutions of:

$$D^2y + PDy + qy = f(x) \qquad f(x) \neq 0$$

is

$$y = C_1 y_1 + C_2 y_2 + y_P$$

y_P is known as the particular solution, and y_1 & y_2 are evaluated by letting $Ly = 0$

HOMOGENEOUS 2nd ORDER DE, W CONSTANT COEFFICENTS

a general form is given by.

$$ay'' + by' + cy = 0$$

then the auxilliary equation is

$$y = e^{mx} \quad \text{or} \quad y = e^{\lambda x}$$

therefore

$$y' = me^{mx} \quad y'' = m^2 e^{mx}$$

and our second order DE becomes.

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

likewise

$$e^{mx}(am^2 + bm + c) = 0$$

since e^{mx} can not be zero.

$$am^2 + bm + c = 0 \quad \Big\} \text{ known as auxilliary equ.}$$

this quadratic equ can have:

CASE 1 - real & unequal distinct roots.

CASE 2 - real but equal roots.

CASE 3 - complex conjugate roots.

for CASE 1, let the roots be m_1 & m_2, hence two solutions are

$$e^{m_1 x} \; ; \; e^{m_2 x}$$

then the complete solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

note: y_1 & y_2 must be linearly independant (in this case they are)

for CASE 2, $m_1 = m_2$, $y_1 = e^{mx}$ the second solution is obtained by differentiating the first solution, w.r.t. m;

$$y_2 = \frac{d}{dm}(e^{mx}) = x e^{mx}$$

therefor the solution is:

$$y = C_1 e^{mx} + C_2 x e^{mx}$$

for CASE 3, where m_1 & m_2 are complex conjiecate pairs.

$$m_1 = \alpha + i\beta$$

$$m_2 = \alpha - i\beta .$$

the solution to our DE equation is

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

we will need to use the identity

$$re^{i\theta} = r\cos\theta + ri\sin\theta$$

then we get.

$$e^{i\beta x} = \cos\beta x + i\sin\beta x$$

$$e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

$$\therefore \quad y_c = e^{\alpha x}\left(C_1 e^{i\beta x} + C_2 e^{-i\beta x}\right)$$

expending the above we get,

$$y_c = e^{\alpha x}\left(C_1\cos\beta x + iC_1\sin\beta x + C_2\cos\beta x - iC_2\sin\beta x\right)$$

$$y_c = e^{\alpha x}\left((C_1 + C_2)\cos\beta x + (C_1 - C_2)i\sin\beta x\right)$$

let $A = C_1 + C_2$
$B = C_1 - C_2$

$$y_c = e^{\alpha x}(A\cos\beta x + Bi\sin\beta x)$$

the above solution can be written into a more compact for as.

$$y_c = e^{\alpha x}\left(E\sin(\beta x + \phi)\right)$$

remember that

$$E\sin(\beta x + \phi) = E\sin\beta x\cos\phi + E\cos\beta x\sin\phi$$

then compare

$$E\sin\beta\cos\phi + E\cos\beta\sin\phi = A\cos\beta x + B\sin\beta$$

we can then see that

$$E\cos\phi = B$$

$$E\sin\phi = A$$

therefore we get

$$\tan\phi = \frac{A}{B} \qquad \text{\&} \qquad E^2 = A^2 + B^2$$

$$\phi = \tan^{-1}\left(\frac{A}{B}\right) \qquad E = \sqrt{A^2 + B^2}$$

EX.

$$2y'' + 5y' - 3y = 0$$

$$(2m^2 + 5m - 3)e^{mx} = 0$$

$$(2m + 1)(m - 3)e^{mx} = 0$$

$$m = -\frac{1}{2},\ 3$$

$$\therefore \quad y = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$$

EX. $y'' - 2y' + y = 0$

$$(m-1)(m-1)e^{mx} = 0$$

$$m = 1$$

$$\therefore \quad y = C_1 e^x + C_2 x e^x$$

EX. $y'' + y' + y = 0$

$$e^{mx}(m^2 + m + 1) = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4AC}}{2A} = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\therefore$

$$y = C_1 e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} + C_2 e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x}$$

$$y = e^{-\frac{1}{2}x}\left(C_1 e^{\frac{\sqrt{3}}{2}ix} + C_2 e^{-\frac{\sqrt{3}}{2}ix}\right)$$

$$e^{\frac{\sqrt{3}}{2}ix} = \cos\frac{\sqrt{3}}{2}x + \sin\frac{\sqrt{3}}{2}x\, i$$

$$e^{-\frac{\sqrt{3}}{2}ix} = \cos\frac{\sqrt{3}}{2}x - i\sin\frac{\sqrt{3}}{2}x$$

therefore we get.

$$y = e^{-\frac{1}{2}x}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_1 i\sin\frac{\sqrt{3}}{2}x + C_2\cos\frac{\sqrt{3}}{2}x - i\sin\frac{\sqrt{3}}{2}x\right.$$

$$= e^{-\frac{1}{2}x}\left((C_1+C_2)\cos\frac{\sqrt{3}}{2}x + (C_1-C_2)\sin\frac{\sqrt{3}}{2}x\right)$$

let $A = C_1 + C_2$
$B = C_1 - C_2$

$$\therefore\ y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

$$y = e^{-\frac{1}{2}x}\left(E\sin\left(\frac{\sqrt{3}}{2}x + \phi\right)\right)$$

where $E = \sqrt{A^2+B^2}$

$\phi = \tan^{-1}\left(\frac{A}{B}\right)$

EX solve the IVP

$$y'' - 4y' + 13y = 0$$

$$y(0) = -1,\quad y'(0) = 2$$

aux eqn.

$$m - 4m + 13 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2}$$

$$= \frac{4}{2} \pm \frac{6}{2}i = 2 \pm 3i$$

$$y_c = e^{2x}(A\cos 3x + B\sin 3x)$$

substitute

$$-1 = A$$

$$y'_c = 2e^{2x}(A\cos 3x + B\sin 3x) + e^{2x}(3B\cos 3x - 3A\sin 3x)$$

$$2 = 2A\cos 3x + 2B\sin 3x + 3B\cos 3x - 3A\sin 3x$$

$$2 = 2A + 3B$$

$$B = \frac{4}{3}$$

$\therefore \quad y_c = e^{2x}\left(-\cos 3x + \frac{4}{3}\sin 3x\right)$

or

$$E = \sqrt{A^2 + B^2} \qquad \phi = \tan^{-1}\left(\frac{-1}{4/3}\right)$$

then

$$y_c = e^{2x}\left(E\sin\left(3x + \phi\right)\right)$$

$$y_c = e^{2x}\left(\frac{5}{3}\sin\left(3x - 0.6435\right)\right)$$

note :

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh = \frac{e^x + e^{-x}}{2}$$

FREQUENTLY OCCURING D.E.

$$y'' + k^2 y = 0$$

$$y'' - k^2 y = 0$$

$$e^{mx}(m^2 + k^2) = 0$$

$$m = \pm ki$$

$$y = C_1 \cos kx + C_2 i \sin kx$$

which is a simple harmonic motion.

$$e^{mx}(m^2 - k^2) = 0$$

$$m = \pm k$$

$$y_c = C_1 e^{kx} + C_2 e^{-kx}$$

we can manipulate C_1, C_2 to get cosh, sinh

$$e^{kx} = \cosh kx + \sinh kx$$

$$e^{-kx} = \cosh kx - \sinh kx$$

$$y_c = (C_1 + C_2)\cosh kx + (C_1 - C_2)\sinh kx$$

HIGHER ORDER DE, w CONSTANT COEFFIENTS.

a general equation of this type will have the form

$$a_n y^n + a_{n-1} y^{n-1} + \ldots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \ldots, a_n$ are constants.

then the general solution where all the λ's are distinct and real, then

$$y_c = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_n e^{\lambda_n x}$$

If all the λ's are the same.

$$y_c = C_1 x^1 e^{\lambda x} + C_2 x^2 e^{\lambda x} + \ldots + C_n x^n e^{\lambda x}$$

EX. $y''' + 3y'' - 4y = 0$

aux equ. $m^3 + 3m^2 - 4 = 0$

by inspection we see $m = 1$ so then we long divide.

$$\begin{array}{r|l} & m^2 + 4m + 4 \\ \hline m - 1 & m^3 + 3m^2 + 0 - 4 \\ & m^3 - m^2 \\ \hline & \quad 4m^2 + 0 - 4 \\ & \quad 4m^2 - 4m \\ \hline & \qquad 4m - 4 \\ & \qquad 4m - 4 \\ \hline & \qquad\quad 0 \end{array}$$

$\therefore m = 1, 2, 2,$

$\therefore y_c = C_1 e^x + C_2 e^{2x} + C_3 x e^x$

EX.

$$y^{(4)} + 2y^{(2)} + y = 0$$

$$(\lambda^2 + 1)^2 = 0$$

$$\lambda_1 = \lambda_3 = i \; ; \; \lambda_2 = \lambda_4 = -i$$

$$y_c = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$$

METHOD OF UNDERTERMINED COEFFICENTS. FOR SOLVING NON-HOMOGENEOUS.

A non-homogeneous DE with constant coeffients of the second order is

$$ay'' + by' + cy = g(x)$$

a, b, c are constants.

To find the solution, we do the following

1. we find y_1, y_2 or y_c
2. we find the particuler solution, y_p of the non-homogeneous DE, and then we find the sum of the following

$$y = y_c + y_p = C_1 y_1 + C_2 y_2 + y_p$$

note that y_1, y_2, y_p must be linearly independ of one another.

METHOD OF UNDETERMINED COEFFIENTS.

is not limited to second order DE, it is however limited to non-homogeneous linear DE. in which

1. The coeffienents are constant

2. $g(x)$ is either a constant or polynomial function; exponentiel function; sine, cosine or sum or product of these functions.

EX. of $g(x)$ are:

$$x^2 - 5x = g(x) \quad \text{(polynomial)}$$
$$x^2 - 5x + e^x = g(x) \quad \text{(poly + exponentiel)}$$
$$\sin(x) = g(x) \quad \text{(sine)}$$
$$e^x \sin(x) + x^2 - 5x = g(x) \quad \text{(all)}$$

note: the method does no apply to function like

$$g(x) = \tan x$$
$$g(x) = \sin^{-1}(x)$$

EX. solve

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

solution

for $y'' + 4y' - 2y = 0$, we assume a solution $e^{\lambda x}$

$$e^{\lambda x}(\lambda^2 + 4\lambda - 2) = 0$$

now we must factor the equation, and it looks as though we will have to use quadratic equation.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$

$$\therefore \quad y_c = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x}$$

now our $g(x) = 2x^2 - 3x + 6$

so let us try to assume a particular solution

$$Ax^2 + Bx + C = y_p.$$

so if y_p is a linear solution to the DE, then it should be able to be substituted in!

$$y_p' = 2Ax + B$$

$$y_p'' = 2A.$$

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

$$(2A) + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

$$(-2A)x^2 + (8A - 2B)x + (2A + 4B - 2C) = 2x^2 - 3x + 6$$

$\therefore \quad -2A = 2$

$\quad A = -1$

$\therefore \quad 8A - 2B = -3$

$\quad -2B = 5$

$\quad B = -5/2$

$\therefore \quad 2A + 4B - 2C = 6$

$\quad -2C = 18$

$\quad C = -9$

$\therefore \quad y_p = -x^2 - \frac{5}{2}x - 9$

then the complete solution is:

$$y = y_c + y_p$$

$$y = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$$

Ex solve for y_p.

$$y'' - y' - 2y = 8e^{3x} \qquad y_c = C_1 e^{2x} + C_2 e^{-x}$$

assume $y_p = Ae^{3x}$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

then substituting we get.

$$(9Ae^{3x}) - (3Ae^{3x}) - 2(Ae^{3x}) = 8e^{3x}$$

$$4Ae^{3x} = 8e^{3x}$$

$$\therefore A = 2 \quad \& \quad y_p = 2e^{3x}$$

then the complete general solution is

$$y = y_c + y_p$$

$$= C_1 e^{2x} + C_2 e^{-x} + 2e^{3x}$$

Ex. find y_p

$$y'' + 2y' = 80 \sin 4x$$

$$y_c = C_1 e^{-2x} + C_2$$

then we assume

$$y_p = A \sin 4x + B \cos 4x$$

$$y_p' = 4A \cos 4x - 4B \sin 4x$$

$$y_p'' = -16A \sin 4x - 16B \cos 4x$$

then substituting

$$(-16A \sin 4x - 16B \cos 4x) + 2(4A \cos 4x - 4B \sin 4x)$$

$$(-16A - 8B) \sin 4x + (-16B + 8A) \cos 4x = 80 \sin 4x$$

then comparing coefficients we get

$$\begin{array}{l|l} -16A - 8B = 80 & \therefore B = -2 \\ +8A - 16B = 0 & \quad A = -4 \\ \quad -40B = 80 & \end{array}$$

then the complete solution becomes

$$y = y_c + y_p$$
$$= C_1 e^{-2} + C_2 - 4\sin 4x - 2\cos 4x.$$

note: sometimes a trial solution does not produce a solution. If fact it produces a zero.

EX. if we choose $y_p = A$

$\therefore \quad y_p' = 0$

$y_p'' = 0$

Ex find y_p of $y' = 5$

$y_p = A \Rightarrow$ doesn't work

but if we assume an Ax

$y_p = Ax$

$y_p' = A$ & we get $y = 5x$

$y_p'' = 0$

EX. Solve the DE given by

$$y'' - 3y = 2e^{3x}$$

find y_c

$$y_c = C_1 e^{0x} + C_2 e^{3x} = C_1 + C_2 e^{3x}$$

now we cannot assume $Ae^{3x} = y_p$ b/c it is not linearly independent from values in y_c, so we must assume

$$y_p = Axe^{3x}$$
$$y_p' = 3Axe^{3x} + Ae^{3x}$$
$$y_p'' = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x}$$

substitude them back in and get.

$$(9Axe^{3x} + 6Ae^{3x}) - 3(3Axe^{3x} + Ae^{3x}) = 2e^{3x}$$

$$(9Ax + 6A - 9Ax - 3A)e^{3x} = 2e^{3x}$$

$$A = 2/3$$

therefore $y = C_1 + C_2 e^{3x} + \frac{2}{3} x e^{3x}$

EX solve the DE give

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

find $y_c = C_1 e^{3x} + C_2 x e^{3x}$

assume $y_p = Ax^2 + Bx + C + Dx^2 e^{3x}$

$y_p' = 2Ax + B + 2Dxe^{3x} + 3Dx^2 e^{3x}$

$y_p'' = 2A + 2De^{3x} + 6Dxe^{3x} + 9Dx^2 e^{3x}$

the substituting.

$= (2A + 2De^{3x} + 6Dxe^{3x} + 9Dx^2 e^{3x}) - 6(2Ax + B + 2Dxe^{3x}$
$+ 3Dx^2 e^{3x}) + 9(Ax^2 + Bx + C + Dx^2 e^{3x})$

$= 2A + 2De^{3x} + 6Dxe^{3x} + 9Dx^2 e^{3x} - 12Ax - 6B - 12Dxe^{3x}$
$- 18Dx^2 e^{3x} + 9Ax^2 + 9Bx + 9C + 9Dx^2 e^{3x}$

BLAH !!

EX. solve $y''' + y' = e^x \cos x$

$y_c = C_1 + C_2 x + C_3 e^{-x}$

then assume $y_p = Ae^x \cos x + Be^x \sin x$

and find derivatives, then substitute.

$y_p''' + y_p' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x$

$\therefore A = -\frac{1}{10} \quad B = \frac{1}{5}$